GOSFORD HIGH SCHOOL

2008 Higher School Certificate

Mathematics Assessment Task #3

June 2008

Time Allowed: *65 minutes plus 5 minutes reading time*

Instructions:

- Show all necessary working.
- Full marks may not be awarded for untidy work, or work which is poorly organised.
- Begin each **new question** on a **new page**.
- Write your **HSC candidate number** at the top of each new page.
- All questions are to be attempted.
- Approved calculators can be used.
- A sheet of indefinite integrals is attached to the back of this question paper.

HSC Candidate Number: ______________________

Question	Mark	Totals	
1	/17	Questions 1 and 3	/31
2	/17	Questions 2 and 4	/29
3	/14		
4	/12		
			/60

Question 1 *(17 marks)*

Marks

a. Differentiate with respect to x:

- i. $\cos x^2$ — 1
- ii. $\tan(\pi - x)$ — 1
- iii. $\dfrac{e^x}{\cos 2x}$ — 2
- iv. $e^{\cos\frac{x}{2}}$ — 2
- v. $\sin^2 3x$ — 2

b. Find $\lim\limits_{x \to 0} \dfrac{\sin 7x}{2x}$ — 1

c. For the function $y = 3 - \cos 2x$:

- i. What is the amplitude? — 1
- ii. What is the period? — 1
- iii. Draw a neat sketch of $y = 3 - \cos 2x$ in the domain $0 \le x \le 2\pi$ — 2

d. Integrate with respect to x:

- i. $\cancel{\cos x \sin^3} x$ $\sin \dfrac{x}{3}$ — 1
- ii. $\cos(3 - 2x)$ — 1
- iii. $\cot 2x$ — 2

Question 2 *(17 marks)*

Marks

a. The first three terms of an arithmetic progression are 9, 13, 17

i. Find the twentieth term of the progression. 2

ii. Which is the first term with a value greater than 1000. 2

iii. Find the sum of 20 terms. 2

b. The first term of a geometric series is $\frac{3}{2}$ and the fourth term is 96.

i. Find the value of the common ratio. 2

ii. Find the ninth term. 2

iii. What is the sum of the first 8 terms? 2

c. i. Express $0.\dot{1}\dot{9}$ as a geometric progression. 1

ii. Using the progression, find the value of $0.\dot{1}\dot{9}$ as a fraction. 1

d. The sum of the first four terms of an arithmetic progression is 36. The sum of the next four terms is 68. Find the first term and the common difference. 3

Question 3 *(14 marks)*

Marks

a. Find the equation of the tangent to the curve y = tan x at the point $(\frac{\pi}{4},1)$ 3

b. Find the area bounded by the x axis and the lines x = 0 and $x = \frac{\pi}{2}$ for the curve $y = \sec^2(\pi - \frac{x}{2})$ 3

c. The curve $y = \sqrt{\sin \pi x}$ where $0 \le x \le \frac{1}{2}$, is rotated about the x axis. Find the volume of the solid of revolution. 3

d. If y = ln(1 + sinx) find $\frac{d^2y}{dx^2}$, and hence show that $\frac{d^2y}{dx^2} + e^{-y} = 0$ 5

Question 4 *(12 marks)*

Marks

a. Stephen saved \$1000 in the first year of his employment. He added \$200 more each subsequent year, therefore saving \$1200 in the next year, and so on. How many years will it take for Stephen to save \$58,000? **3**

b. A plant is observed over a period of time. Its initial height is 30cm. It grows 5cm during the first week of observation. Each successive week the growth is 80% of the previous week's growth. Assuming this pattern of growth continues, calculate the plant's ultimate height. **2**

c. Susan is employed at an initial salary of \$18,900 per year. At the end of each year she receives an increase of \$x per year for the following year. At the end of 10 years she found that her total earnings were eight times the amount that she earned in the tenth year. Find the value of x. **3**

d. i. How many terms of the progression 64, 48, 36, ... must be added to give a sum which exceeds 250? **3**

ii. Show that ~~a~~ the sum of any finite number of terms of this sequence will never exceed 256. **1**

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \; x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \; x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \; x > 0$

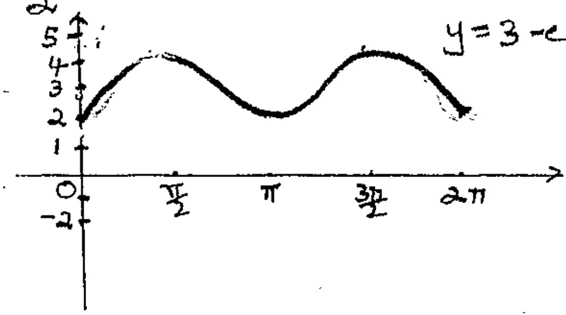
Year 12 Assess #3 (1)

June 2008 – Solutions

Q1 c) i) 1

* ii) $\frac{2\pi}{2} = \pi$

iii)



NB. Change of order:
Q1. c, b, a, d.
Q4. d, a, b, c.

b) $\lim_{x \to 0} \frac{\sin 7x}{2x \times \frac{7}{2}} \times \frac{7}{2}$

$= \lim_{x \to 0} \frac{7}{2} \frac{\sin 7x}{7x}$

$= \frac{7}{2}$

* a). i) $\frac{d}{dx}(\cos x^2) = -2x \sin x^2$

ii) $\frac{d}{dx}\tan(\pi - x) = -\sec^2(\pi - x)$

iii) $\frac{d}{dx}\frac{e^x}{\cos 2x} = \frac{e^x \cdot \cos 2x - e^x \times -2\sin 2x}{\cos^2 2x}$

$= \frac{e^x(\cos 2x + 2\sin 2x)}{\cos^2 2x}$

iv) $\frac{d}{dx} e^{\cos \frac{x}{2}} = -\frac{1}{2}\sin\frac{x}{2} \cdot e^{\cos \frac{x}{2}}$

v) $\frac{d}{dx}\sin^2 3x = 2\sin 3x \times 3\cos 3x$

$= 6\sin 3x \cos 3x$

d). i) $\int \cos x \sin^3 x \, dx = \frac{\sin^4 x}{4} + c$

ii) $\int \cos(3 - 2x)\,dx = -\frac{1}{2}\sin(3 - 2x) + c$

iii) $\int \cot 2x \, dx = \int \frac{\cos 2x}{\sin 2x}\,dx$

$= \frac{1}{2}\log \sin 2x + c$

Q2. a) $a=9, d=4, n=20$

i) $T_n = a+(n-1)d$

$T_{20} = 9 + 19 \times 4$

$= 9 + 76$

$= 85$

ii) $9+(n-1)4 > 1000$

$5+4n > 1000$

$4n > 995$

$n > 248\frac{3}{4}$

∴ The term is the 249^{th}

iii) $S_n = \frac{n}{2}(a+l)$

$= 10(9+85)$

$= 940$

b) $a = \frac{3}{2}$ $\quad T_4 = ar^3$

$= 96.$

i) $\frac{3}{2}r^3 = 96$

$r^3 = 64$

$\therefore r = 4$

∴ common ratio is 4

ii) $T_n = ar^{n-1}$

$T_9 = \frac{3}{2} \times 4^8$

$= 98{,}304$

iii) $S_n = \frac{a(r^n-1)}{r-1}$

$S_8 = \frac{3}{2}\frac{(4^8-1)}{4-1}$

$= \frac{3}{2}\left(\frac{65536-1}{3}\right)$

$= 32{,}767.5$

c) i) $0.\dot{1}\dot{9} = \frac{19}{100} + \frac{19}{10{,}000} + \frac{19}{1{,}000{,}000} + \ldots$

ii) $\lim S = \frac{a}{1-r}$ where $a = \frac{19}{100}$ and $r = \frac{1}{100}$

$= \frac{\frac{19}{100}}{1-\frac{1}{100}}$

$= \frac{19}{99}$

d) $S_n = \frac{n}{2}[2a+(n-1)d]$

$36 = \frac{4}{2}[2a+3d]$

$36 = 2(2a+3d)$

ie, $2a+3d = 18$ (1)

$104 = \frac{8}{2}[2a+7d]$

$104 = 4(2a+7d)$

ie. $2a+7d = 26$ (2)

(2) − (1)

$4d = 8$

$d = 2$

When $d=2$, $2a+6 = 18$

$2a = 12$

$a = 6.$

∴ First term is 6 and common difference is 2

Question 3

a) $y = \tan x$

$\frac{dy}{dx} = \sec^2 x$

when $x = \frac{\pi}{4}$

$\frac{dy}{dx} = \sec^2 \frac{\pi}{4}$

$= 2$

$m_{tang.} = 2$

Eqn. of tangent:

$y - y_1 = m(x - x_1)$

$y - 1 = 2(x - \frac{\pi}{4})$

$y - 1 = 2x - \frac{\pi}{2}$

$2x - y + 1 - \frac{\pi}{2} = 0$

b). $A = \int_0^{\frac{\pi}{2}} \sec^2\left(\pi - \frac{x}{2}\right) dx$

$= -2\left[\tan\left(\pi - \frac{x}{2}\right)\right]_0^{\frac{\pi}{2}}$

$= -2\left[\tan\left(\pi - \frac{\pi}{4}\right) - \tan(\pi - 0)\right]$

$= -2\left[\tan \frac{3\pi}{4} - 0\right]$

$= -2 \times -1$

$= 2$

$\therefore$ area is 2 units^2

c) $y = \sqrt{\sin \pi x}$

$y^2 = \sin \pi x$

$V = \pi \int_0^{\frac{1}{2}} \sin \pi x \, dx$

$= -\frac{1}{\pi} \times \pi \left[\cos \pi x\right]_0^{\frac{1}{2}}$

$= -\left[\cos \frac{\pi}{2} - \cos 0\right)$

$= -\left[0 - 1\right]$

$= 1.$

$\therefore$ Volume is 1 unit^3

d). $y = \ln(1 + \sin x)$

$\frac{dy}{dx} = \frac{\cos x}{1 + \sin x}$

$\frac{d^2y}{dx^2} = \frac{-\sin x(1 + \sin x) - \cos x . \cos x}{(1 + \sin x)^2}$

$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2}$

$= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1 + \sin x)^2}$

$= \frac{-\sin x - 1}{(1 + \sin x)^2}$

$= \frac{-(\sin x + 1)}{(1 + \sin x)^2}$

$= \frac{-1}{1 + \sin x}$

Now $e^{-y} = \frac{1}{e^y}$

$= \frac{1}{e^{\ln(1 + \sin x)}}$

$= \frac{1}{1 + \sin x}$

$\therefore \frac{d^2y}{dx^2} + e^{-y} = \frac{-1}{1 + \sin x} + \frac{1}{1 + \sin x}$

$= 0$ as required

Question 4

*a) i) $a = 64,\ r = \frac{48}{64}$
$= \frac{3}{4}$

$S_n = \frac{a(1-r^n)}{1-r}$

$S_n = \frac{64(1-\frac{3}{4}^n)}{1-\frac{3}{4}}$

$= 256(1-\frac{3}{4}^n)$

Now for $256(1-(\frac{3}{4})^n) = 250$

$1-(\frac{3}{4})^n = \frac{250}{256}$

$-(\frac{3}{4})^n = -\frac{3}{128}$

$(\frac{3}{4})^n = \frac{3}{128}$

$n \ln(\frac{3}{4}) = \ln(\frac{3}{128})$

$n = \ln(\frac{3}{128}) \div \ln(\frac{3}{4})$

$n = 13.0471042$

$\therefore$ The first term to be added so that the sum exceeds 250 is the 14^{th} term

ii) $\lim S = \frac{a}{1-r}$ $\quad |r| < 1$

$= \frac{64}{1-\frac{3}{4}}$

$= 256$

$\therefore$ Sum never exceeds 256

b) $1000 + 1200 + 1400 + \ldots$

$a = 1000,\ d = 200 \quad n = ?$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$58{,}000 = \frac{n}{2}[2000 + (n-1)200]$

$116{,}000 = 2000n + 200n^2 - 200n \quad (\div 100)$

$1160 = 20n + 2n^2 - 2n$

ie. $n^2 + 9n - 580 = 0$

$(n+29)(n-20) = 0$

$\therefore n = -29$ (neg does not apply) or $n = 20$

It will take 20 years to save \$58,000

b) Height = 30cm.

$a = 5,\ r = 80\%$

$= \frac{4}{5}$

Now $|r| < 1$

$\therefore \lim S = \frac{a}{1-r}$

$= \frac{5}{1-\frac{4}{5}}$

$= 25$

$\therefore$ Plant's ultimate height is 55 cm

d) $a = \$18{,}900 \quad d = x$

$T_{10} = 18{,}900 + 9d$

$S_{10} = \frac{10}{2}[2 \times 18{,}900 + 9d]$

$= 5[37800 + 9d]$

Now $5(37{,}800 + 9d) = 8(18{,}900 + 9d)$

$189{,}000 + 45d = 151200 + 72d$

$37800 = 27d$

$\therefore d = 1400$

Hence the value of x if 1400